STUDY OF THE TRANSVERSE VIBRATION OF THE ELASTIC-PLASTIC STRING UNDER DIFFERENT PLASTICITY CONDITIONS

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ABSTRACT. Theoretical work on the wave propagation in an elastic-plastic string struck transversely at its middle point is discussed in this paper graphically. The only basic assumption is that the tension of the string is some known non-linear function of strain This means that the phase velocity of the transverse wave changes from point to point as the pulse is propagated through such a string which ultimately becomes assymetrical in shape. The main object of this paper is to explain graphically :

(i) variations in displacements with time,
(ii) variations in pressure with time,
(iii) time of collision under different plasticity conditions, into three different sections.

INTRODUCTION

Before the discussion of the problem under consideration something must be said about the elastic-plastic behaviour of the string employed in the present issue. The foundation of the theory of plasticity has not yet been firmly established and the various survey papers about the subject differ from one another not only in scope but also in the points of view of their respective authors. In the case of a perfectly elastic string vibrating under transverse impact the stress-strain law is provided by a linear relation which is independent of time. It may be noted in this connection that any deviation of the assumption about this linearly in the stress to strain relation will introduce plasticity in the material of the string. In the present theory strain is neither linearly dependent on strain nor does it depend upon the strain-rate but unlike the case of a perfectly flexible string the tension is assumed to be a known non-linear function of strain. The important contribution of this assumption is that the phase velocity of the string due to transverse impact does not remain constant as the pulse is propagated along the string, but depends upon strain and changes from point to point of it. Thus the velocities at different points are different functions of strain. Naturally the velocity gradients at different points of the string are also different functions of strain and the measure of the change in velocity gradient at the struck point is evidently a measure of the plasticity of the string.

For the purpose of a thorough and a much better investigation of the above theory some theoretical graphs are drawn and the various interesting results coming out of them are found to agree well with the earlier theories of the subject matter under discussion. In this paper special attention is given to the discussion of the graphical results as stated in the abstract in three different sections:

EXPLANATIONS OF THE SYMBOLS USED

l = Length of the string $= a+b$.

a = Shorter segment of the string.

b = Longer segment of the string.

s = Variable measured along length of the string fixed at $s = o$ and $s = l$.

t = Variable time.

y_a = Displacement of the struck point.

ρ = linear density of the string.

m = Mass of the hammer.

ϵ = Variable strain at any point of the string.

$c_1(\epsilon)$ = Velocity of the transverse wave motion of the string in the portion $o < s < a$.

$c_2(\epsilon)$ = Velocity of transverse wave motion of the string in the portion $a < s < l$.

$c_a(\epsilon)$ = Velocity of transverse wave motion at the struck point.

ν_0 = Velocity of impact.

P = Pressure exerted by the hammer.

$$\psi(\epsilon) = \left[\left(\frac{dc_2}{ds}\right)_{s=a} - \left(\frac{dc_2}{ds}\right)_{s=a}\right]$$

$$q = \frac{2\rho c_a}{m}.$$

$$r = 9\psi(\epsilon)$$

$$\theta_a = \frac{2a}{c_a}.$$

It has already been stated in the abstract that the paper proposes to find out displacement and pressure fluctuations at the struck point of the string. In doing so computations are made with the help of some numerical datas as :

$l = 96$ cm, $\quad m = 25$ gms, $\quad \rho = 1$ gm/cm, $\quad c_a = 3000$ cm/sec.

$\nu_a = 40$ cm/sec, $\quad \theta = \frac{2l}{c_a} = \cdot 064$ sec, $\quad q = \frac{2\rho c_a}{m} = 300.$

and $\quad \sqrt{q^2-4r} = 10\sqrt{900-12\psi(\epsilon)}$

TIME DISPLACEMENT VARIATIONS AT THE STRUCK POINT

The expression for the displacement at the struck point as obtained by Ghose *et al* (1965) in an earlier publication during the 1st epoch is,

$$y_a = \frac{v_0 A}{\beta+\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]$$

where (α, β) are given by

$$[\alpha, \beta] = 150 \pm 5\sqrt{900-12\psi(\epsilon)}$$

It can be easily seen that the nature of the values of (α, β) depend upon the discriminant of (2) i.e., $\sqrt{900-12\psi(\epsilon)}$. Thus the values of (α, β) will be either all real distinct or real equal or else imaginary depending upon the values of $\psi(\epsilon)$. The discussion is therefore restricted to these three different cases that may arise. When $q^2 > 4r$ i.e., when $\psi(\epsilon) < 75$

$$y_a = \frac{2v_0}{\sqrt{q^2-4r}}\, e^{-\frac{1}{2}qt} \sinh\left\{\frac{1}{2}\sqrt{q^2-4r}\right\} t$$

when $q^2 = 4r$, i.e., when $\psi(\epsilon) = 75$,

$$y_a = v_0 t\, e^{-\frac{1}{2}qt}$$

when $q^2 < 4r$ i.e., when $\psi(\epsilon) > 75$,

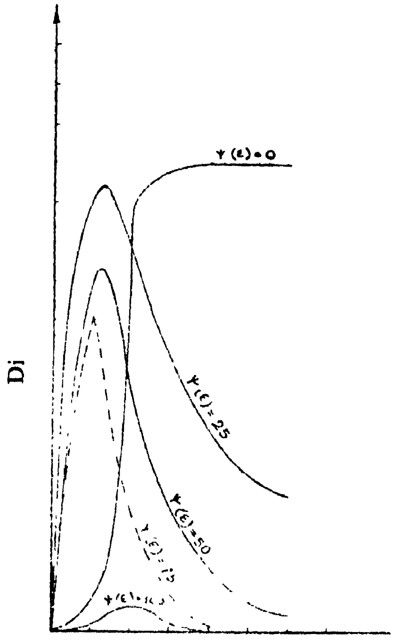
$$y_a = \frac{2v_0}{\sqrt{4r-q^2}}\, e^{-\frac{1}{2}qt} \sin\left\{\frac{1}{2}\sqrt{4r-q^2}\right\} t$$

Fig. 1 represents the complete behaviour of time displacement variations for the case $\psi(\epsilon) \lesseqgtr 75$.

The curve for $\psi(\epsilon) = 0$, i.e. when the string is elastic shows that the displacement increases with time exponentially and ultimately becomes steady at a finite value.

Curves for $0 < \psi(\epsilon) \leqslant 75$ which is the critical value of $\psi(\epsilon)$ $[\psi(\epsilon) \geqslant 75]$ show a distinct feature analogous to the damped vibration in string. Here the maxima of the displacements decrease as $\psi(\epsilon)$ increases. But the rate of fall of displacement increases progressively with $\psi(\epsilon)$. This is clearly due to increased damping associated with the increased plasticity of the material

The case for $\psi(s) > 75$ makes the time-displacement curve damped oscillatory. The amplitude of vibration of this curve though at first increasing is much less pronounced in this case than in other cases of $\psi(\epsilon)$. It then remains almost constant during the first epoch as it is clear from the graph itself. This shows



$C_a t$

Fig. 1.

that immediately after impact the pulse propagates along the string with more or less a constant velocity. The fall of displacement is however much more slow in this case due to increased plasticity of the string as the case should be At large value of plasticity it is associated with large damping. The amplitude is therefore very small and the curve resembles a highly damped motion.

The theoretical time-displacement graphs obtained by the present author reveals the fact that the displacements gradually diminish due to increased plasticity of the string, a conclusion quite analogous to that derived by Kolsky (1960) in the case of thin bars which are visco-elastic in nature.

PRESSURE-TIME VARIATION AT THE STRUCK POINT

The expressions for pressure at different epochs exerted by the hammer on the string as derived by the author in an earlier publication (Ghosh, 1965) are as follows :

During the interval, $\quad 0 < t < \theta_a$

$$P_1 = \frac{mv_0}{(q^2-4r)^{\frac{1}{2}}} [\alpha^2 e^{-\alpha t} - \beta^2 e^{-\beta t}] \tag{1}$$

During, $\theta_a < t < 2\theta_a$,

$$P_2 = P_1 + \frac{2mv_0q}{q^2-4r}\,[\alpha^2(2+A-\alpha t_1)e^{-\alpha t_1}+\beta^2(2-A-\beta t_1)\,e^{-\beta t_1}] \tag{2}$$

where,

$$[\alpha, \beta] = \tfrac{1}{2}\,[q \pm \sqrt{q^2-4q\psi(\epsilon)}] \tag{3}$$

It may be observed that the r.h.s. of (3) actually explains the nature of the roots (α, β). The only undefined quantity on the r.h.s. of (3) is $\psi(\epsilon)$ which is termed as the 'representative of plasticity' in the string and capable of assuming any arbitraty value. Naturally the values of (α, β) may be either real unequal or real equal, or else imaginary subject to the 3 conditions $q^2 \gtreqless 4q\psi(c)$ i.e., $\psi(\epsilon) \lesseqgtr 75$.

The main object of this section is to study the pressure-time variations under different plasticity conditions i.e., corresponding to different values of $\psi(\epsilon)$. It is therefore necessary to define the expressions for pressure at different epochs suitably relative to various values for $\psi(\epsilon)$.

Thus for values of $q^2 \gtreqless 4q\psi(\epsilon)$ i.e., $\psi(\epsilon)$ 75 the pressure expression during the different epochs are given,

During, $0 < t < \theta_a$, when $q^2 > 4q\psi(c)$ i.e., when $\psi(c) < 75$

$$P_1 = \frac{mv_0e^{-\frac{1}{2}qt}}{(q^2-4r)^{\frac{1}{2}}}\left[(q^2-2r)\sinh\frac{(q^2-4r)^{\frac{1}{2}}}{2}\,t - q(q^2-4r)^{\frac{1}{2}}\cosh\frac{(q^2-4r)^{\frac{1}{2}}}{2}\,t\right]$$

Similarly when, $q^2 = 4r$, i.e., $\psi(\epsilon) = 75$.

$$P_1 = mv_0q\psi(\epsilon)t\,e^{-\frac{1}{2}qt}$$

Similarly when, $q^2 < 4q\psi(\epsilon)$ i.e., $\psi(\epsilon) > 75$

$$P_1 = \frac{2mv_0q}{(4r-q^2)^{\frac{1}{2}}}\,\psi(\epsilon)e^{-\frac{1}{2}qt}\sin\left\{\frac{1}{2}(4r-q^2)^{\frac{1}{2}}t+\tan^{-1}\frac{q(4r-q^2)^{\frac{1}{2}}}{2r-q^2}\right\}$$

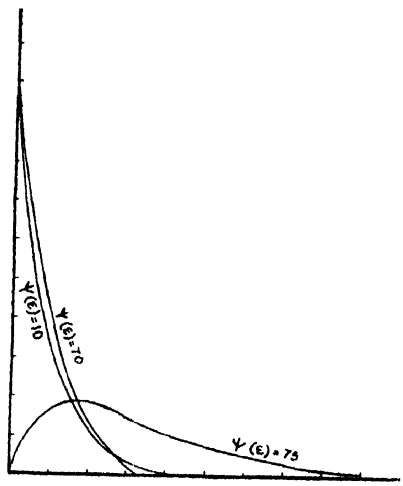
It will be observed later that pressure falls to zero during the 1st epoch in all the cases excepting the critical one and so the expressions for pressure in higher epochs are not written here.

With these expressions for pressure as a function of time a few graphs are drawn under various plastcity conditions and the different interesting conclusions derived from them agree well with the earlier theoretical results about the matter.

Figs. 2 and 3 correspond to the pressure time variations under different values of $\psi(\epsilon)$.

Time

Fig. 2



Time

Fig. 3

Fig. 2 represents the pressure-time curve for $\psi(s) = 0$. The case corresponds to that of perfectly elastic string. Here the pressure which is very large at the beginning falls to a minimum, becomes high as a fresh new wave is generated at the beginning of the 2nd epoch. The behavioiur of the sting in this case is quite similar to that derived by Ghose (1952) in the case of a perfectly flexible string. Fig. 3 is a complete picture of the pressure-time variations due to the increased plasticity of the string.

By studying the pressure time curves for various values of $\psi(\epsilon)$ it is found that for values $\psi(\epsilon) < 75$ i.e., $q < 4r$, the pressure suddently jumps to a value kv_0c at $t = 0$ and then falls exponentially to zero within the first epoch with comparatively little change in nature. But the duration of contact diminishes as $\psi(\epsilon)$, the representative of plasticity, increases. This means that the medium becomes more and more dispersive as well as dissipative in nature. The above remarks receive a strong support from the experimental results of Ghosh *et al* (1965) who, in the case of a thin bar, has shown that the pressure terminates during the first epoch when it is struck by a light and soft (or plastic) load.

The curve for $\psi(\epsilon) = 75$ i.e., for $q^2 = 4r$ is critical. By studying the pressure time variation in this case it is found that the amplitude of the stress pulse is much diminished showing that the response of this critical plasticitycondition on the pressurepulse is so marked that the shape of the pressure curve is changed altogether. The progressive rise of the pressure pulse is rather smooth and the rate of fall is more slow showing no tendency of the pressure being terminated within the first epoch.

The curve for $\psi(\epsilon) > 75$ i.e., shows that when the material of string is more plastic, stress is not generated in the string by impact shown by the negative values of pressure. The energy of impact is dispersed so quickly that the string undergoes very small displacement at the struck point as shown by the time displacement curve for $\psi(\epsilon) = 100$.

Phase angle versus $\psi(\epsilon)$:

It has been observed that when $q^2 < 4r$, the pressure equation becomes damped oscillatory. This result is in agreement with the case of a light and soft load striking a flexible string transversely. The values of $\psi(\epsilon) > 75$ i.e., large values of plasticity are responsible for the initiation of a type of waves through the material that the stress developed in the specimen due to the propagation of pulse is no longer in phase with it. The stress becomes more and more out of phase with the pulse as the value of $\psi(\epsilon)$ increases. This feature is depicted in fig 4, in which the variation of δ with $\psi(\epsilon)$ is shown For large values of $\psi(\epsilon)$,

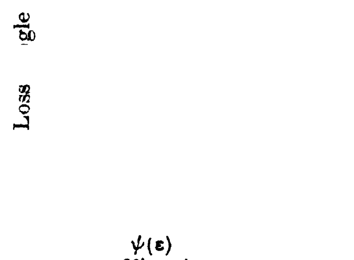


Fig. 4

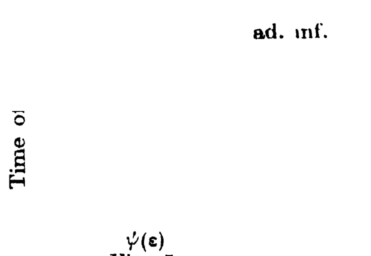


Fig. 5

δ is necessarily large which means that the stress needs a longer time to rise. Here,

$$\delta = \tan^{-1}\frac{q(4r-q^2)^{\frac{1}{2}}}{2r-q^2}$$

TIME OF COLLISION UNDER DIFFERENT PLASTICITY CONDITIONS

When the impacting load strikes the string, it first moves in the forward direction, then momentarily comes to rest and then begins to move in the opposite direction. The duration for which the string remains in contact with the moving load is defined to be the time of collision.

The time of collision plays a very important part which explains the actual acoustical behaviour of the string vibrating in any mode. The amplitude of vibration at different harmonics depends upon the pressure imparted to the string, as well as, on the time of collision for which the pressure acts from the beginning. The expression for pressure at the struck point has already been derived by the author in a previous publication. The purpose of this section is to examine graphically the time of collision under different plasticity conditions, i.e., $\psi(\epsilon)$, the representative of plasticity, assuming different arbitrary values.

The time of collision for any particular epoch can be found algebraically to be the lowest positive root obtained by solving the pressure equation at the struck point to zero i.e., $P_n(t) = 0$. This method is employed when it becomes difficult to obtain time of collision graphically, usually at higher epochs.

Fig. 5 represents graphically how the nature of the times of collision between the load and string changes as the plasticity increases more and more.

The time of collision is comparatively large in the case of an elastic string i.e., corresponding to value of $\psi(\epsilon) = 0$. It then falls suddenly and then attains almost a steady state for values of $0 < \psi(\epsilon) < 75$. The portion of the graph for this range of values of $\psi(\epsilon)$ is almost a straight line whose slope gradually diminishes until the critical stage is reached. When the critical value is attained by $\psi(\epsilon)$ i.e., when $\psi(\epsilon) = 75$ the time of collision jumps to infinity showing thereby that the load remains in contact with the string and moves with it.

The discussions made in the above three sections depict the actual dynamical conditions of the string struck transversely at its middle point. The corresponding conditions when it is struck near one end will be published in a subsequent issue of the journal.

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